J Math Chem (2011) 49:843–858
DOI 10.1007/s10910-010-9782-3

ORIGINAL PAPER

Cubic polyhedral Ramanujan graphs with face size no larger than six

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Received: 5 August 2010 / Accepted: 20 November 2010 / Published online: 16 December 2010

Abstract A k-regular graph is *Ramanujan* if its second largest eigenvalue (by magnitude) has magnitude less than or equal to $2\sqrt{k-1}$. Exhaustive search up to the bound of 138 vertices, derived from Spielman and Teng's work on graph partitioning, finds all cubic polyhedral Ramanujan graphs with positive curvature, i.e., with face sizes no larger than 6. Of all such polyhedra, those with face sizes 5 or 6, i.e., fullerenes, give the largest known examples of cubic Ramanujan polyhedra (with 84 vertices). We also consider the notions of negative and positive Ramanujan graphs, as those without eigenvalues in the respective open intervals $(-k, -2\sqrt{k-1})$ and $(2\sqrt{k-1}, k)$. Our results give the full list of positive cubic polyhedral Ramanujan graphs with positive curvature but for negative Ramanujan graphs we have only a finiteness theorem and a conjectured complete list.

Keywords Graph theory · Eigenvalues · Ramanujan graphs · Polyhedra

1 Introduction

A k-regular graph on n vertices is a Ramanujan graph if the second largest (in absolute value) eigenvalue λ of its adjacency matrix obeys

The first author is supported by the Croatian Ministry of Science, Education and Sport under contract 098-0982705-2707.

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$$|\lambda| \leq 2\sqrt{k-1}.$$

Ramanujan graphs are of interest in several contexts, including the theory of expander graphs for network design [1–3]. There is specific interest in planar Ramanujan graphs [4], and the program GRAFFITI [5] has produced some conjectures for the cubic polyhedral graphs known as fullerenes. A fullerene is a cubic polyhedral graph with face sizes five and six only. The fullerenes are of interest in chemistry as the skeletons of carbon cage molecules such as C_{60} [6]. Fullerenes that are Ramanujan graphs have been called *ramafullerenes* [5] and by analogy we will use the term (*cubic*) *ramapolyhedra* for (cubic) polyhedral Ramanujan graphs. Computational search [7] of small fullerenes, motivated by the GRAFFITI results, found ramafullerenes from 20 to 84 vertices. It was conjectured on the basis of explicit search up to 100 vertices that there were no ramafullerenes with more than 84 vertices. The two 84-vertex ramafullerenes are apparently still the largest known planar Ramanujan graphs; see [4] and "About the cover" in the same issue of Notices of the AMS.

The separator theorem of Lipton and Tarjan [8] and its improvements by Alon, Seymour and Thomas [9] provide an upper bound of 875 vertices on cubic Ramanujan graphs, but this is well beyond the capabilities of published methods for fullerene enumeration [10]. In the present work, we take advantage of a recent result of Spielman and Teng [11], which brings the bound for cubic Ramanujan graphs down to 138 vertices, to (i) prove the completeness of the list of ramafullerenes in [7], and (ii) count the Ramanujan graphs within all 19 classes of cubic polyhedra with positive curvature (i.e., those with no face of size larger than six). The 19 classes arise from the different characteristic triples (p_3, p_4, p_5) compatible with the Euler relation for cubic graphs, $\sum_r (6-r)p_r = 12$, where p_r is the number of faces of size r. By Eberhard's theorem (See [12]), all 19 classes are realisable for some values of p_6. The largest Ramanujan graphs within each class are illustrated.

A graph with no eigenvalue in $(-k, -2\sqrt{k-1})$, $(2\sqrt{k-1}, k)$ is called negative, respectively positive Ramanujan. The upper bound of 138 applies as well to positive cubic Ramanujan graphs. For negative Ramanujan graphs, we prove finiteness of the list as part of a general theorem that says that there are only a finite number of (p_3, p_4, p_5)-graphs whose eigenvalues are not contained in an interval (a, b) with $-3 \leq a < b \leq 3$. So, we have the complete list of positive Ramanujan graphs but only a conjectured list of negative Ramanujan graphs.

2 Bounds

The n adjacency eigenvalues of a k-regular graph, arranged in non-increasing order are $+k = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -k$. The *separator* of a k-regular graph G is $\text{sep}(G) = k - \lambda_2$. A Ramanujan graph ('ramagraph') is a finite regular graph of degree k for which all eigenvalues (other than $\pm k$) have magnitude at most $2\sqrt{k-1}$. Ramanujan k-regular graphs are therefore those with one eigenvalue $\lambda_1 = +k$, possibly one eigenvalue $\lambda_n = -k$ (if and only if the graph is bipartite), and all other λ_i such that $|\lambda_i| \leq 2\sqrt{k-1}$. Following [7], we call here *positive* (respectively *negative*) ramagraphs those that have no eigenvalues in the open interval $(2\sqrt{k-1}, k)$, respectively $(-k, -2\sqrt{k-1})$. Clearly, a graph is Ramanujan if and only it is both a positive and a negative ramagraph.

The first result used is the *Lipton Tarjan theorem* (see [8,13]), which has been applied in many algorithmic contexts [14]. The version presented below is an improvement due to Alon, Seymour and Thomas [9].

Theorem 2.1 [9] *If G is a planar graph, then there exists a cycle F of length n_3, that separates G into two patches P_1 and P_2 with n_1 and n_2 vertices, satisfying*

$$n_1 + \frac{n_3}{2} \leq \frac{2n}{3}, \quad n_2 + \frac{n_3}{2} \leq \frac{2n}{3} \quad \text{and} \quad n_3 \leq \frac{3\sqrt{2n}}{2}.$$

Theorem 2.2 [15] *If G is a k-regular graph and $\{B, C\}$ is a partition of its vertex set, then*

$$e(B, C) \geq \text{sep}(G) \frac{|B|.|C|}{n}$$

with $e(B, C)$ the number of edges between B and C.

From this we derive:

Theorem 2.3 *If G is a positive cubic ramapolyhedron then G has at most* 875 *vertices.*

Proof Take a partition from 2.1. Every vertex of the cycle F is adjacent to one vertex in P_1 or P_2. Denote by n_i the number of vertices of F adjacent to P_i. There exists i_0, say $i_0 = 1$, such that $n_1 \leq n_3/2$. The partition chosen is then $B = P_1$, $C = F \cup P_2$. From Theorem 2.2, we have

$$e(B, C) = n_1 \leq \frac{3\sqrt{2n}}{4}.$$

From the equations of Theorem 2.1 we have

$$|B| \leq \frac{2n}{3} \quad \text{and} \quad |C| = n_2 + n_3 \leq \frac{2n}{3} + \frac{3\sqrt{2n}}{4}.$$

Since $|C| = n - |B|$, by writing $f(x) = x(n - x)$ we obtain

$$|B|.|C| = f(|B|) \geq \min\left\{ f\left(\frac{2n}{3}\right), f\left(\frac{n}{3} - \frac{3\sqrt{2n}}{4}\right)\right\} = f\left(\frac{n}{3} - 3\frac{3\sqrt{2n}}{4}\right).$$

By using $\text{sep}(G) \geq 3 - 2\sqrt{2}$ and Theorem 2.2, we obtain after a simple search the upper bound of 875 vertices. □

The above result implies finiteness of the set of cubic Ramanujan planar graphs; finiteness without an explicit bound was already proved in the *minuteman* section of [5]. In fact, a tighter upper bound for cubic polyhedra can be derived from the work of Spielman and Teng on graph partitions [11].

Theorem 2.4 *If G is a positive cubic ramapolyhedron then G has at most* 138 *vertices.*

Proof The fact that G is a positive cubic ramapolyhedron implies $\mathrm{sep}(G) \geq 3 - 2\sqrt{2}$. However, in [11] it is proved that planar graphs with degree k and n vertices have

$$\mathrm{sep}(G) \leq \frac{8k}{n}$$

hence, $n \leq 139.8$ and the bound follows. □

In fact, it turns out that this upper bound is still rather pessimistic. The derivation in [11] relies on a circle-packing argument on the 2-dimensional sphere, i.e., that there is a packing of disks on the sphere and so the density is bounded by 1.

Below we give a general finiteness result that applies to any interval of the spectrum at the price of not giving an explicit bound on the maximal number of vertices and being restricted to graphs of positive curvature. By (6^3) we denote the infinite tessellation of the plane by hexagons. A *p-patch* is an hexagon with p-rings of hexagons added around it. A *g-nanotube* is a quotient of (6^3) by the group generated by one translation, with g being the girth of the obtained map. A (p, g)*-nanotube* is a section of length p of a g-nanotube.

Lemma 2.5 *For any characteristic triple* (p_3, p_4, p_5) *and number p there exists a number* N_0 *such that any* 3*-valent plane graph of characteristic* (p_3, p_4, p_5) *with more than* N_0 *vertices contains either a p-patch or a* (p, g)*-nanotube.*

Proof Take a graph G with characteristic triple (p_3, p_4, p_5). Suppose that every hexagonal face is within r faces of a face with less than 6 sides. Then the graph is covered by the $p_3 + p_4 + p_5$ disks of radius r centered at those sub-hexagonal faces. Hence the total number of faces is bounded and the number of vertices as well. So, if the number of vertices is large enough, then G contains an hexagon F such that the faces in the set $\mathcal{H}$ of all faces within distance r of F are hexagons. If $\mathcal{H}$ does not self intersect then it is a patch and we are done. Otherwise, this self-intersection gives a path P of faces that splits G into two components C_1 and C_2. Denote by $Curv_i = \sum_{F \in C)i} (6 - l(F))$ the sum of the curvatures of the faces in C_i. If $Curv_1 < 6$ then $Curv_2 > 6$ and this means that the girth of $\mathcal{H}$ varies and increases when one moves towards the faces of non-zero curvature in C_2. So G contains a patch also in this case. If $Curv_1 = Curv_2 = 6$ then $\mathcal{H}$ is a nanotube itself and we are done. □

There is a way to describe the classes of characteristic (p_3, p_4, p_5) by means of $p_3 + p_4 + p_5 - 2$ complex parameters [16]. The parameterizations are not unique and are acted on by a group. The quotient of the set of possible parameters by the group generated by the equivalence transformation (called the *monodromy group* in [17]) is a non-compact topological space, which corresponds to the cubic polyhedra of characteristic (p_3, p_4, p_5). The directions in which it is non-compact correspond exactly to the partitions of the set of faces of non-zero curvature into two sets with sum of curvature 6 ([16], p. 533). One could also prove the above lemma by using this theory.

Theorem 2.6 *If $I = [a, b]$ with $a < b$ is an interval contained in $[-3, 3]$ and (p_3, p_4, p_5) is a characteristic triple then there is a finite number of graphs of positive curvature having no eigenvalues in I.*

Proof In [18] the spectrum of toroidal fullerenes is determined. Denote by $A_{(6^3)}$ the infinite adjacency matrix of (6^3). Since the spectrum of $A_{(6^3)}$ is entirely *essential* (see [19]) and equal to $[-3, 3]$ [18] for every $\lambda \in [-3, 3]$ there exists an eigenvector f_λ of $A_{(6^3)}$ with $|f_\lambda(v)| = 1$ for any vertex v. Let us now take one hexagon of (6^3) and a surrounding p-patch G_p. Denote by $f_{\lambda,p}$ the function defined by taking

$$f_{\lambda,p}(v) = \begin{cases} f_\lambda(v) & \text{if } v \in G_p, \\ 0 & \text{if } v \notin G_p. \end{cases}$$

The L^2-norm of $f_{\lambda,p}$ will be proportional to the area of G_p and thus of order p^2, while the L^2-norm of $A_{(6^3)} f_{\lambda,p} - \lambda f_{\lambda,p}$ is estimated by the perimeter of G_p and hence of order p. We now define $g_{\lambda,p} = f_{\lambda,p}/\|f_{\lambda,p}\|$ and find

$$\|g_{\lambda,p}\| = 1 \quad \text{and} \quad \lim_{p\to\infty} \|A_{(6^3)} g_{\lambda,p} - \lambda f_{\lambda,p}\| = 0.$$

For nanotubes we proceed in a similar way. Their spectrum is determined in [20] and instead of a p-patch we take a (p, g)-nanotube. The norm of $f_{\lambda,p}$ will be proportional to p and $\|A_{(6^3)} g_{\lambda,p} - \lambda f_{\lambda,p}\|$ will be bounded. Now let us take an interval $I = [a, b]$ and $\lambda \in I$. If A is a symmetric $n \times n$-matrix with eigenvalues $(\lambda_p)_{1\le p\le n}$ having no eigenvalue in I then, if $\|f\| = 1$, we have

$$\begin{aligned} \|Af - \lambda f\|^2 &= \langle (A - \lambda Id)^2 f, f\rangle \\ &= \sum_{p=1}^{n} f_p^2 (\lambda_p - \lambda)^2 \\ &\ge \min\left\{(a-\lambda)^2, (b-\lambda)^2\right\} \sum_{p=1}^{n} f_p^2 \\ &\ge \min\left\{(a-\lambda)^2, (b-\lambda)^2\right\}. \end{aligned}$$

Thus, if we can find vectors with $\|Af - \lambda f\|$ sufficiently small, then we will have proved that there is an eigenvalue in I.

In summary, by Lemma 2.5 if the number of vertices of a graph G of characteristic (p_3, p_4, p_5) is large enough, then it contains a p-patch or a (p, g)-nanotube. But if G has such a structure for p large then for the adjacency matrix A of G there is a vector f of norm 1 with $\|Af - \lambda f\|$ small which is clearly impossible by the above estimation. □

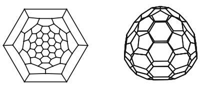
In particular, there are only a finite number of (p_3, p_4, p_5)-graphs that are negative Ramanujan, although we have not obtained an explicit bound. Based on the computations it seems reasonable to expect that we have found the complete list of negative

Ramanujan cubic polyhedra, with faces of size at most 6, although we have no proof of this.

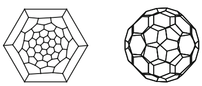
3 Enumeration of graphs

By the previous theorems, it is possible to complete the enumeration of (p_3, p_4, p_5)-ramagraphs by checking adjacency spectra of all cubic polyhedra with positive polyhedra up to the limit of 138 vertices. The graphs were constructed using the CPF program, for which the algorithm is explained in [21]. For (4, 0, 0)-graphs, there is an explicit description of the possible eigenvalues [18]. For other graphs, we had to use numerical techniques to compute the eigenvalues. We used the Jacobi method which is known for its high accuracy [22], Chapter 14. However since we want exact results, this is not sufficient a priori. We choose a generous error limit of $\epsilon = 10^{-5}$ and if all eigenvalues (excluding ± 3) λ satisfy $|\lambda| \leq 2\sqrt{2} - \epsilon$ we conclude that the graph is Ramanujan. If one eigenvalue (excluding ± 3) has $|\lambda| > 2\sqrt{2} + \epsilon$, then we conclude that the graph is not Ramanujan. If neither condition is satisfied then we resolve the case by an exact algebraic computation using determinants, which fortunately we do not have to do very often.

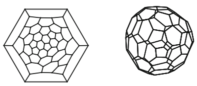
In fact, although the number is finite, there are many 3-valent Ramanujan graphs. As a consequence we tabulate some overall counts but draw only the maximal examples in each class. We also give the number of graphs in each class in Tables 1, 2 and 3. We did not find any (p_3, p_4, p_5)-graph whose largest or smallest eigenvalue (excluding ± 3) is $2\sqrt{2}$ or $-2\sqrt{2}$. Note that a 3-regular 20-vertex but non-planar graph with these eigenvalues is found in [23], which answers a question of [24].



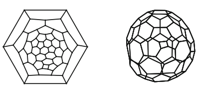
$(0, 0, 12)$, T_d, 84



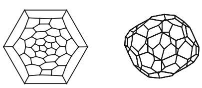
$(0, 0, 12)$, D_{2d}, 84



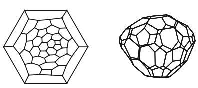
$(0, 1, 10)$, C_1, 74



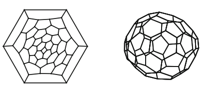
$(0, 1, 10)$, C_1, 74



$(0, 2, 8)$, C_2, 70



$(0, 2, 8)$, C_1, 70



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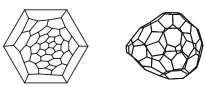

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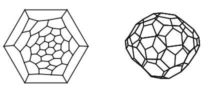

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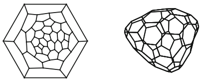
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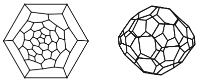
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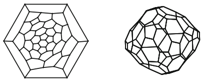
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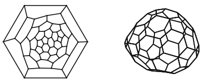
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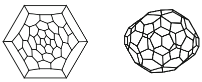
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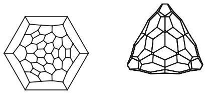
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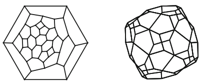
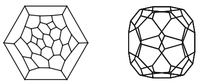
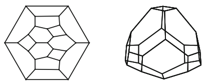
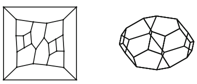

$(0,4,4)$, S_4, 52


$(0,4,4)$, D_2, 52


$(0,5,2)$, C_{2v}, 26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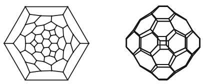
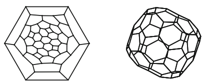

$(0,6,0)$, O_h, 72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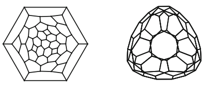
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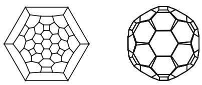
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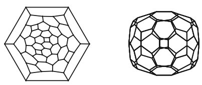
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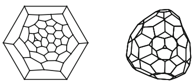
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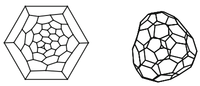
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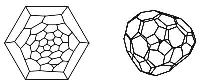
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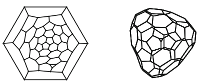
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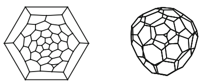
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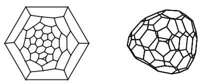
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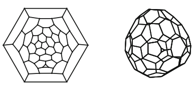
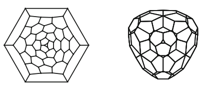
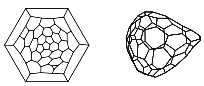
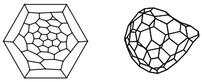
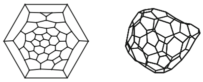

$(1,0,9)$, C_1, 72


$(1,0,9)$, C_{3v}, 72

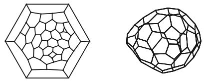

$(1,1,7)$, C_1, 70



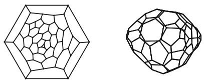
$(1, 1, 7)$, C_1, 70



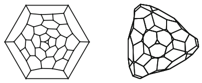
$(1, 1, 7)$, C_1, 70



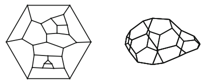
$(1, 2, 5)$, C_1, 68



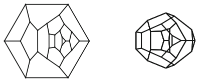
$(1, 2, 5)$, C_1, 68



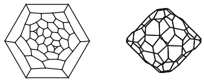
$(1, 3, 3)$, C_3, 60



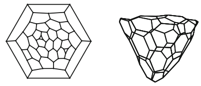
$(1, 4, 1)$, C_1, 30



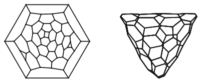
$(1, 4, 1)$, C_s, 30



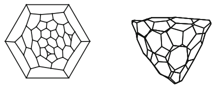
$(2, 0, 6)$, C_2, 72



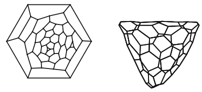
$(2, 1, 4)$, C_1, 66



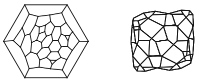
$(2, 1, 4)$, C_1, 66



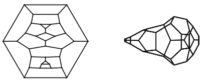
$(2, 1, 4)$, C_1, 66



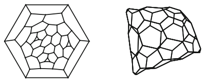
$(2, 1, 4)$, C_1, 66



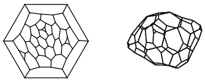
$(2, 2, 2)$, C_2, 58



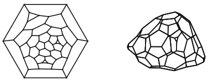
$(2, 3, 0)$, C_s, 34



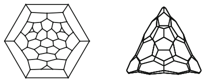
$(3, 0, 3)$, C_1, 62



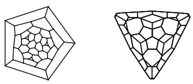
$(3, 0, 3)$, C_1, 62



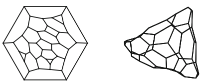
$(3, 0, 3)$, C_1, 62



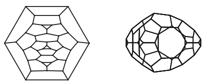
$(3, 0, 3)$, C_s, 62



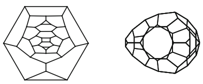
$(3, 0, 3)$, C_{3v}, 62



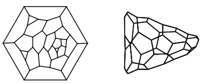
$(3, 1, 1)$, C_1, 44



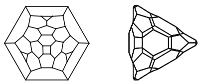
$(3, 1, 1)$, C_s, 44



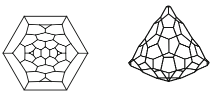
$(3, 1, 1)$, C_s, 44



$(3, 1, 1)$, C_1, 44



$(3, 1, 1)$, C_s, 44



$(4, 0, 0)$, T_d, 64

Table 1 Numbers of negative cubic ramapolyhedra by number of vertices and by characteristic triple

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
26	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	1	0	0	0	2	1	0	1	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	1	1	5	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	3	0	0	4	2	1	0	0	0
36	0	0	0	0	1	0	0	0	1	6	7	1	7	6	6	0	0	2	0
38	0	0	1	1	1	0	0	0	5	16	7	1	10	21	7	0	0	0	0
40	0	0	1	3	1	0	0	1	17	28	10	0	28	12	18	0	0	1	0
42	0	0	1	13	4	0	0	1	28	36	15	0	36	28	7	0	3	0	0
44	0	0	19	19	4	0	0	6	80	50	12	0	76	37	12	0	1	3	0
46	0	5	35	22	2	0	0	21	144	88	10	0	74	62	5	0	7	0	0
48	1	16	87	31	2	0	0	30	255	92	10	0	151	71	13	0	4	4	0
50	1	27	162	24	5	0	0	61	390	100	5	0	165	78	12	0	7	0	0
52	11	116	329	28	0	0	0	126	588	104	9	0	254	59	10	0	4	1	0
54	22	219	459	24	0	0	0	213	734	85	4	0	242	79	3	0	15	0	0
56	60	443	683	20	1	0	0	276	970	99	1	0	360	62	4	0	0	3	0
58	113	785	848	6	0	0	0	564	1,135	58	0	0	314	69	2	0	18	0	0
60	313	1,188	976	9	0	0	0	734	1,291	39	0	0	461	46	1	0	4	1	0
62	549	1,632	1,113	10	0	0	0	1,132	1,319	62	0	0	362	52	1	0	20	0	0
64	1,098	2,503	1,148	2	0	0	0	1,459	1,281	33	0	0	443	25	0	0	3	0	0
66	1,863	3,136	1,051	7	0	0	0	1,840	1,301	18	0	0	346	18	0	0	7	0	0

Table 1 continued

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
68	3,018	3,656	936	0	0	0	0	1,998	1061	11	0	0	379	16	0	0	3	0	0
70	4,592	3,793	707	0	0	0	0	2,372	849	6	0	0	250	15	0	0	5	0	0
72	6,958	3,767	504	0	0	0	0	2,298	697	0	0	0	277	1	0	0	0	0	0
74	8,823	3,261	373	0	0	0	0	2,371	491	0	0	0	150	2	0	0	1	0	0
76	11,202	2,692	254	0	0	0	0	2,125	335	0	0	0	145	1	0	0	0	0	0
78	13,041	1,978	170	0	0	0	0	1,809	234	0	0	0	99	0	0	0	2	0	0
80	14,265	1,497	100	0	0	0	0	1,519	143	0	0	0	68	0	0	0	0	0	0
82	14,508	1,001	41	0	0	0	0	13,00	79	0	0	0	47	0	0	0	0	0	0
84	14,577	720	27	0	0	0	0	925	51	0	0	0	34	0	0	0	0	0	0
86	14,229	443	18	0	0	0	0	720	50	0	0	0	21	0	0	0	0	0	0
88	13,657	248	5	0	0	0	0	465	8	0	0	0	12	0	0	0	0	0	0
90	12,433	123	5	0	0	0	0	296	3	0	0	0	7	0	0	0	0	0	0
92	11,100	63	0	0	0	0	0	158	1	0	0	0	8	0	0	0	0	0	0
94	9,688	29	0	0	0	0	0	121	2	0	0	0	1	0	0	0	0	0	0
96	8,349	11	1	0	0	0	0	68	1	0	0	0	7	0	0	0	0	0	0
98	6,721	3	1	0	0	0	0	28	0	0	0	0	1	0	0	0	0	0	0
100	5,615	2	0	0	0	0	0	13	0	0	0	0	0	0	0	0	0	0	0
102	4,197	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0
104	2,969	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
106	2,150	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
108	1,737	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
110	1,301	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1 continued

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
112	960	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
114	564	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
116	330	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
118	224	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
120	141	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
122	51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
124	33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
126	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
128	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
130	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
132	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For $n > 132$, all counts are zero

Table 2 Numbers of positive cubic ramapolyhedra by number of vertices and by characteristic triple

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
16	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
18	0	0	0	0	1	2	0	0	0	0	0	3	0	0	0	1	0	0	0
20	0	0	0	0	1	2	0	0	0	0	1	2	0	0	1	0	0	0	0
22	0	0	0	0	4	5	0	0	0	0	4	2	0	0	3	1	0	0	0
24	0	0	0	0	13	4	0	0	0	0	8	4	0	0	6	0	0	0	0
26	0	0	0	2	16	8	0	0	0	1	10	6	0	0	2	1	0	0	0
28	0	0	1	8	33	8	0	0	0	1	27	6	0	1	8	1	0	1	0
30	0	0	1	17	48	16	0	0	0	13	32	11	0	3	8	5	0	0	0
32	0	0	3	34	74	21	0	0	1	21	52	10	0	2	23	1	0	0	0
34	0	0	9	76	116	24	0	0	0	51	87	12	0	14	23	3	1	0	0
36	0	0	28	135	172	27	0	0	2	73	127	17	0	9	35	0	0	1	0
38	0	0	34	250	204	44	0	0	3	159	135	18	0	29	24	2	2	0	0
40	0	0	96	378	312	34	0	0	13	204	188	13	1	25	67	3	0	1	0
42	0	0	160	628	377	54	0	0	27	360	219	30	2	66	34	6	2	0	0
44	0	1	334	865	472	61	0	0	68	446	279	19	5	56	71	1	2	7	0
46	0	13	520	1,248	538	66	0	0	99	667	301	21	5	96	43	6	5	0	0
48	0	37	979	1,612	720	72	0	2	238	759	362	27	18	107	73	2	1	5	0
50	0	128	1,484	2,177	745	85	0	20	318	1,064	336	28	10	152	54	2	10	0	0
52	0	344	2,310	2,610	871	70	0	24	528	1,033	405	18	20	97	74	1	3	7	0
54	1	771	3,117	3,116	923	75	0	9	544	1,218	330	27	5	166	47	3	4	0	0
56	1	1,264	4,355	3,496	994	85	0	2	814	1,230	351	16	8	92	84	2	0	3	0

Table 2 continued

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
58	0	2,060	5,180	4,009	974	75	0	5	748	1,177	360	17	6	136	39	3	2	0	0
60	3	2,637	6,161	3,830	1,030	60	0	6	858	975	248	19	11	68	41	0	3	3	0
62	5	3,583	6,733	4,048	873	80	0	13	775	916	216	9	2	58	21	1	4	0	0
64	25	4,423	7,107	3,744	880	35	0	7	749	625	186	10	5	31	9	1	0	0	0
66	8	5,028	6,315	3,123	617	50	0	6	480	444	104	6	0	31	9	0	0	0	0
68	27	5,081	5,612	2,327	453	13	0	2	318	208	46	2	0	2	5	0	1	0	0
70	25	4,598	4,169	1,821	302	31	0	0	61	89	21	0	0	2	0	0	1	0	0
72	28	3,187	2,646	891	286	5	0	0	11	4	5	0	0	0	0	0	0	0	0
74	11	1,854	1,121	368	44	16	0	0	0	0	0	0	0	0	0	0	0	0	0
76	19	965	510	122	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
78	4	224	130	33	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80	3	21	16	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
82	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For $n > 82$, all counts are zero

Table 3 Numbers of cubic ramapolyhedra by number of vertices and by characteristic triple

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1
10	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	1	0	0
12	0	0	0	0	1	0	1	0	0	0	1	1	1	1	2	0	0	1	2
14	0	0	0	1	1	1	1	0	0	1	1	1	0	2	3	1	2	0	0
16	0	0	1	1	3	1	1	0	0	3	3	1	1	1	8	1	0	1	3
18	0	0	1	2	3	0	1	0	1	4	7	0	2	6	6	2	2	0	0
20	1	0	2	5	9	0	3	0	1	9	10	1	5	9	9	0	1	4	1
22	0	1	3	11	7	0	1	1	4	16	16	1	3	14	12	2	4	0	0
24	1	1	11	14	12	0	3	0	10	24	18	2	13	14	18	2	1	2	2
26	1	3	13	32	13	2	3	2	13	47	22	1	7	34	15	2	9	0	0
28	2	5	29	42	22	0	3	3	28	65	32	0	26	27	27	0	2	6	2
30	3	10	49	61	23	0	2	5	46	91	35	2	22	57	14	1	11	0	0
32	6	20	88	92	26	0	6	8	73	137	32	0	41	55	23	0	1	5	3
34	6	37	124	127	12	0	3	18	110	167	38	0	40	83	17	1	12	0	0
36	15	57	208	144	21	0	6	20	184	202	26	0	80	84	15	0	5	4	2
38	17	109	299	184	11	0	6	44	235	255	20	0	60	120	20	0	17	0	0
40	40	163	443	211	9	0	5	51	355	279	38	0	115	93	18	0	5	3	1
42	45	278	593	196	8	0	4	99	446	303	17	0	86	144	15	0	24	0	0
44	89	404	806	211	6	0	11	114	592	315	12	0	140	111	9	0	4	5	1
46	116	634	987	214	2	0	4	190	674	290	6	0	110	133	13	0	21	0	0
48	197	895	1,184	170	0	0	9	224	810	254	6	0	152	85	3	0	4	0	3

Table 3 continued

n	(0, 0, 12)	(0, 1, 10)	(0, 2, 8)	(0, 3, 6)	(0, 4, 4)	(0, 5, 2)	(0, 6, 0)	(1, 0, 9)	(1, 1, 7)	(1, 2, 5)	(1, 3, 3)	(1, 4, 1)	(2, 0, 6)	(2, 1, 4)	(2, 2, 2)	(2, 3, 0)	(3, 0, 3)	(3, 1, 1)	(4, 0, 0)
50	268	1,250	1,313	148	0	0	8	315	799	212	0	0	100	97	1	0	16	0	0
52	424	1,503	1,323	114	2	0	7	324	784	181	0	0	137	75	2	0	6	0	1
54	554	1,832	1,321	74	0	0	5	451	796	126	0	0	95	74	1	0	14	0	0
56	853	2,028	1,166	56	0	0	7	466	675	104	0	0	106	55	0	0	7	0	1
58	1,076	2,211	946	45	0	0	5	518	538	88	0	0	59	28	1	0	14	0	0
60	1,456	2,301	743	36	0	0	4	482	452	43	1	0	67	36	0	0	1	0	1
62	1,772	2,293	536	13	0	0	5	475	260	29	0	0	32	11	0	0	5	0	0
64	2,180	1,736	226	1	0	0	3	322	144	9	0	0	23	5	0	0	0	0	1
66	2,276	1,264	78	0	0	0	0	269	65	2	0	0	13	4	0	0	0	0	0
68	2,527	714	27	0	0	0	2	120	25	2	0	0	8	0	0	0	0	0	0
70	2,292	281	10	1	0	0	0	33	3	0	0	0	0	0	0	0	0	0	0
72	1,723	72	0	0	0	0	5	8	0	0	0	0	1	0	0	0	0	0	0
74	1,300	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
76	745	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
78	156	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80	25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
82	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
84	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For $n > 84$, all counts are zero

Acknowledgments We thank Jack Graver for help with the proof of Lemma 2.5 and Nathan Linial for pointing us to [11].

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